Problem Set 2

This second problem set is designed to make you a master of inductive proofs. It starts off with some simpler problems intended to acclimate you to an inductive climate and concludes with some pretty impressive results. I hope that you have as much fun working through these problems as we did designing them.

Start this problem set early. It contains six problems (plus one checkpoint question, one survey question, and one extra-credit question), several of which require a fair amount of thought. I would suggest reading through this problem set at least once as soon as you get it to get a sense of what it covers.

As much as you possibly can, please try to work on this problem set individually. That said, if you do work with others, please be sure to cite who you are working with and on what problems. For more details, see the section on the honor code in the course information handout.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. You should, at the very least, state what type of proof you are using, and (if proceeding by contradiction, contrapositive, or induction) state exactly what it is that you are trying to show. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

If you are asked to prove something by induction, you may use either weak induction or strong induction. You should state your base case before you prove it, and should state what the inductive hypothesis is before you prove the inductive step.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 150 possible points and ten questions. It is weighted at 7% of your total grade. The earlier questions serve as a warm-up for the later problems, so do be aware that the difficulty of the problems does increase over the course of this problem set.

Good luck, and have fun!

Checkpoint Questions Due Monday, April 16 at 2:15PM

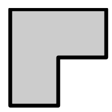
Remaining Questions Due Friday, April 20 at 2:15 PM

Checkpoint Question: Tiling with Triominoes* **(25 Points if Submitted)**

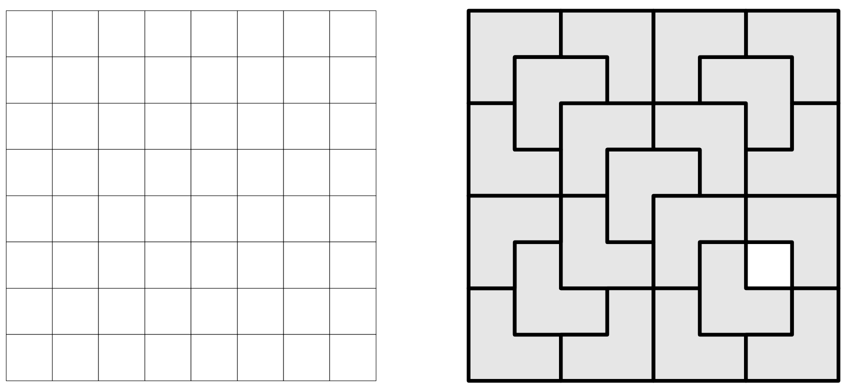
Write your solutions to the following problems and submit them by this Monday, April 16th at the start of class. These problems will be graded based on whether or not you submit it, rather than the correctness of your solutions. We will try to get these problems returned to you with feedback on your proof style this Wednesday, April 18th. Submission instructions are on the last page of this problem set.

Please make the best effort you can when solving these problems. We want the feedback we give you on your solutions to be as useful as possible, so the more time and effort you put into them, the better we'll be able to comment on your proof style and technique. **Note that this question has three parts.**

Suppose that we are given a set of **right triominoes**, blocks with this shape:



Suppose that we are given a square grid of size $2^n \times 2^n$ and want to *tile* it with right triominoes by covering the grid with triominoes such that all triominoes are completely on the grid and no triominoes overlap. Here's an attempt to cover an 8×8 grid with triominoes, which fails because not all squares in the grid are covered:



It turns out that it is impossible to completely tile the $2^n \times 2^n$ figure with these triominoes because there are 4^n squares in a $2^n \times 2^n$ figure, but any figure that can be tiled with triominoes must have a multiple of three squares in it. However, 4^n is not a multiple of three. Interestingly, though, if we remove some square from the $2^n \times 2^n$ grid, then there are a total of $4^n - 1$ squares to cover, which *is* a multiple of three.

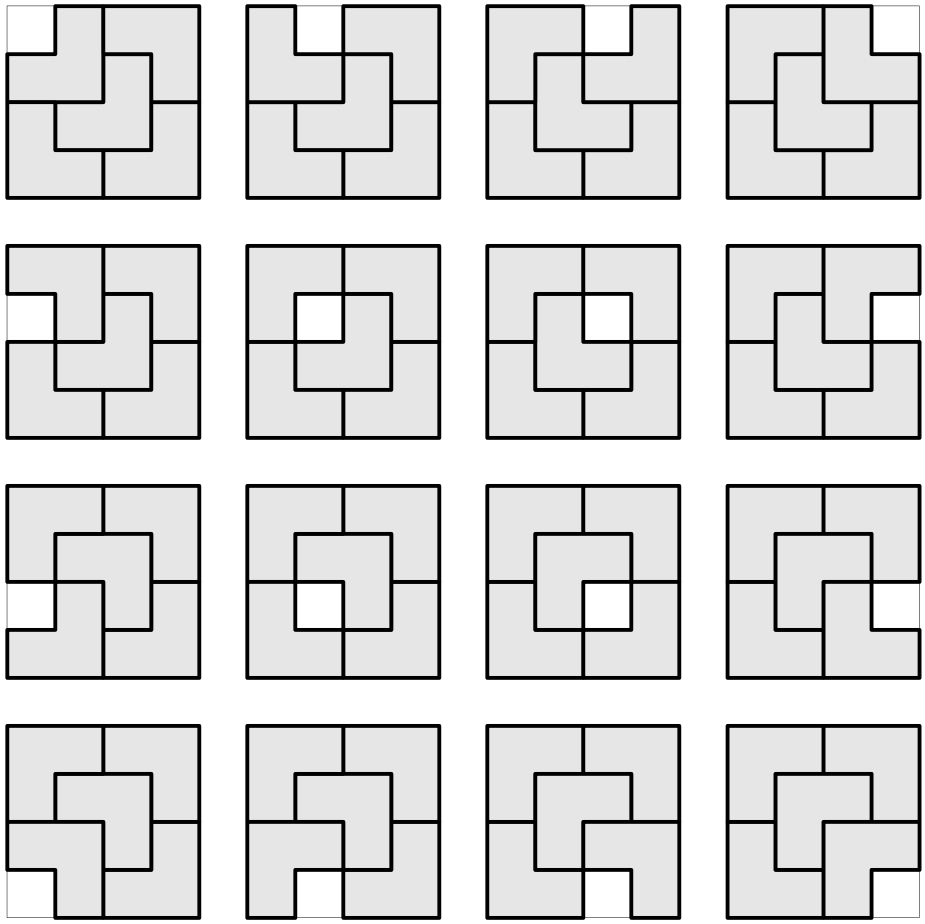
i. Prove, by induction on n, that $4^n - 1$ is a multiple of three.

ii. It is *necessary* that a figure have a multiple of three squares in it to be tiled with right triominoes, in that if it does not contain a multiple of three squares there cannot possibly be a way to fully tile it. However, it is not true that it is *sufficient* that if a figure has a multiple of three squares in it that it must can be tiled with right triominoes. Give an example of a shape that contains an multiple of three squares but cannot be tiled by right triominoes.

(continued on next page)

* I originally heard this classic problem from David Gries of Cornell University. The terminology used here is borrowed from *Discrete Math and its Applications, Sixth Edition* by Kenneth Rosen.

Amazingly, it turns out that it is always possible to tile any $2^n \times 2^n$ grid that's missing exactly one square with right triominoes. It doesn't matter what n is or which square is removed; there is always a solution to the problem. For example, here are all the ways to tile a 4 × 4 grid that has a square missing:



iii. Prove that any $2^n \times 2^n$ grid with one square removed can be tiled by right triominoes.

The remainder of these problems should be completed and returned by Friday, April 20 at the start of class.

Problem One: Telescoping Sums (8 points)

Consider a sequence of $n + 1$ numbers $x_0, x_1, x_2, \ldots, x_n$, where $n \geq 0$. Prove, by induction, that

$$\sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0$$

Sums of this form are called *telescoping sums* and have many applications in computer science.

Problem Two: Nim (16 points)

Nim is a family of games played by two players. Each game works by maintaining several piles of stones. Players alternate taking turns. In each turn, the player removes any (nonzero) number of stones from any one pile of their choice. If it's a player's turn and no stones are left in any of the piles, then the player loses the game.

Prove that if the game is played with two piles of stones, each of which begins with the same number of stones, then the second player can always win the game.

Problem Three: Contract Rummy (20 points)

Contract rummy is a card game for any number of players (usually between three and five) in which players are dealt a hand of cards and, through several iterations of drawing and discarding cards, need to accumulate *sets* and *sequences*. A set is a collection of three cards of the same value, and a sequence is a collection of four cards of the same suit that are in ascending order. The game proceeds in multiple rounds, in each of which the players need to accumulate a different number of sets and sequences. The rounds are:

- Two sets (six cards)
- One set, one sequence (seven cards)
- Two sequences (eight cards)
- Three sets (nine cards)
- Two sets and a sequence (ten cards)
- One set and two sequences (eleven cards)
- Three sequences (twelve cards)

Notice that in each round, the requirements are such that the number of cards required increases by one. It's interesting that it's always possible to do this, since the total number of cards must be made using just combinations of three cards and four cards.

Prove, by induction, that any natural number greater than or equal to six can be written as $3x + 4y$ for natural numbers x and y (remember that 0 is a natural number).

Problem Four: Fun with Sums (20 Points)

Using induction, we saw how to prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

This allows us to replace a sum with a much simpler formula. In this problem, you will do the same for two other sums.

i. Find a simple formula for the sum of the first n odd natural numbers, then prove by induction that your formula is correct. Your formula should not contain a summation (either as a Σ or as a sum containing an ellipsis).

ii. Find a simple formula for the sum of the first n even natural numbers, then prove by induction that your formula is correct. Remember that 0 is the first even natural number. Your formula should not contain a summation (either as a Σ or as a sum containing an ellipsis).

iii. In class, before we proved the result about the sum of the first n positive natural numbers, we drew a picture that illustrated why the result ought to be true. For either (i) or (ii), provide a justification for your answer without using induction. You don't need to provide a formal proof here; simply explain the intuition behind the result through some other means.

Problem Five: Repeated Squaring (24 points)

In many applications in computer science, especially cryptography, it is important to compute exponents efficiently. For example, the RSA public-key encryption system, widely used in secure communication, relies on computing huge powers of large numbers. Fortunately, there is a fast algorithm called *repeated squaring* for computing x^y in the special case where y is a natural number.

The repeated squaring algorithm is based on the following function *RS*:

$$RS(x, y) = \begin{cases} 1 & \text{if } y = 0 \\ RS(x, y/2)^2 & \text{if } y \text{ is even and } y > 0 \\ x \cdot RS(x, (y-1)/2)^2 & \text{if } y \text{ is odd and } y > 0 \end{cases}$$

For example, we could compute 2^{10} using *RS*(2, 10) follows:

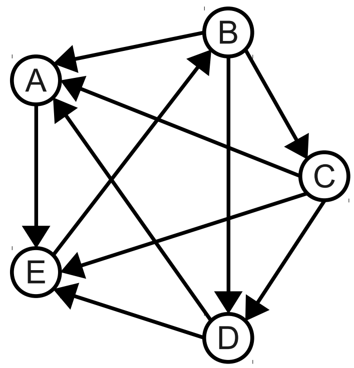
In order to compute $RS(2, 10)$, we need to compute $RS(2, 5)^2$.
    In order to compute $RS(2, 5)$, we need to compute $2 \cdot RS(2, 2)^2$.
        In order to compute $RS(2, 2)$, we need to compute $RS(2, 1)^2$.
            In order to compute $RS(2, 1)$, we need to compute $2 \cdot RS(2, 0)^2$.
                By definition, $RS(2, 0) = 1$
            so $RS(2, 1) = 2 \cdot RS(2, 0)^2 = 2 \cdot 1^2 = 2$.
        so $RS(2, 2) = RS(2, 1)^2 = 2^2 = 4$.
    so $RS(2, 5) = 2 \cdot RS(2, 2)^2 = 2 \cdot 4^2 = 32$.
so $RS(2, 10) = RS(2, 5)^2 = 32^2 = 1024$.

The reason that the *RS* function is interesting is that it can be computed much, much faster than simply multiplying x by itself y times. Since *RS* is defined recursively in terms of *RS* with the y term cut in half, *RS* can be evaluated using approximately $\log_2 y$ multiplications. (You don't need to prove this).

Prove that for any $x \in \mathbb{R}$ and any $y \in \mathbb{N}$, $RS(x, y) = x^y$. Note that since x is a real number, you **cannot** prove this by induction on x. However, you may use induction on y.

Problem Six: Tournament Graphs (32 points)

A **tournament** is a contest among $n > 0$ players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). A **tournament graph** is a graph representing the result of a tournament, where each node corresponds to a player and each edge (u, v) means that player u won her game against player v. For example, here is a simple tournament graph for five players:



A **tournament winner** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won against that player. For example, in the above graph, B, C, and E are tournament winners.

Prove that every tournament graph has a tournament winner.

Problem Seven: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. How hard did you find this problem set? How long did it take you to finish?

ii. Does that seem unreasonably difficult or time-consuming for a five-unit class?

iii. Did you attend Monday's problem session? If so, did you find it useful?

iv. How is the pace of this course so far? Too slow? Too fast? Just right?

v. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-spr1112-submissions@lists.stanford.edu) with the string "[PS2]" somewhere in the subject line.

If you are an SCPD student, we would strongly prefer that you submit solutions via email, especially for the checkpoint problems, so that we can get your solution graded and returned as quickly as possible. Please contact us if this will be a problem.

Extra Credit Problem: Egyptian Fractions

The Fibonacci sequence is named after Leonardo Fibonacci, an eleventh-century Italian mathematician who is credited with introducing Hindu-Arabic numerals (the number system we use today) to Europe in his book *Liber Abaci*. This book also contained an early description of the Fibonacci sequence, from which the sequence takes its name.

In lecture, we saw a surprising connection between Fibonacci numbers and *continued fractions*, an system for writing out rational numbers. Interestingly, *Liber Abaci* also described a separate notation for fractions called *Egyptian fractions*, a method for writing out fractions that has been employed since ancient times.* An Egyptian Fraction is a sum of <u>distinct</u> fractions whose numerators are all one (the so-called *unit fractions*). For example:

$$\frac{1}{3}=\frac{1}{2}+\frac{1}{6} \qquad \frac{2}{15}=\frac{1}{10}+\frac{1}{30}$$

$$\frac{7}{15}=\frac{1}{3}+\frac{1}{8}+\frac{1}{120} \qquad \frac{2}{85}=\frac{1}{51}+\frac{1}{255}$$

One way of finding an Egyptian fraction representation of a rational number is to use a *greedy algorithm* that works by finding the largest unit fraction at any point that can be subtracted out from the rational number. For example, to compute the fraction for 42 / 137, we would start off by noting that 1 / 4 is the largest unit fraction less than 42 / 137. We then say that

$$\frac{42}{137}=\frac{1}{4}+\left(\frac{42}{137}-\frac{1}{4}\right)=\frac{1}{4}+\frac{31}{548}$$

We then repeat this process by finding the largest unit fraction less than 31 / 548 and subtracting it out. This number is 1/18, so we get

$$\frac{42}{137}=\frac{1}{4}+\left(\frac{42}{137}-\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{18}+\left(\frac{31}{548}-\frac{1}{18}\right)=\frac{1}{4}+\frac{1}{18}+\frac{5}{4932}$$

* There is archaeological evidence (the Rhind papyrus) that shows that the ancient Egyptians were using Egyptian fractions over three thousand years ago.

The largest unit fraction we can subtract from 5 / 4932 is 1 / 987:

$$\frac{42}{137}=\frac{1}{4}+\frac{1}{18}+\left(\frac{5}{4932}-\frac{1}{987}\right)=\frac{1}{4}+\frac{1}{18}+\frac{1}{987}+\frac{1}{1622628}$$

And at this point we're done, because the leftover fraction is itself a unit fraction.

Prove that the greedy algorithm for continued fractions always terminates for any rational number in the range (0, 1) and always produces a valid Egyptian fraction. That is, the sum of the unit fractions should be the original number, and no unit fraction should be repeated. This shows that every rational number in the range (0, 1) has at least one Egyptian fraction representation.